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Impurity magnetisation on the Ising chain in a transverse field through a weak-coupling expansion

Loïc Turban

Laboratoire de Physique du Solide†, Université de Nancy I CO 140, F-54037, Nancy Cedex, France

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Abstract. Ladder and chain perturbations in the two-dimensional Ising model are studied using a weak-coupling expansion for the local magnetisation on the corresponding quantum Ising chains in a transverse field. Using a duality transformation, these models are shown to display different critical exponents for their local order and disorder parameters.

1. Introduction

Since the work of Bariev (1979) and McCoy and Perk (1980) a linear perturbation in the spin interaction energy of the two-dimensional (2D) Ising model is known to result in a non-universal behaviour of the local magnetisation and the two-spin correlation function. This could have been anticipated on the basis of the Kadanoff–Wegner (1971) or Bray–Moore (1977) marginality analysis.

Bariev (1979) studied the ladder perturbation (figure 1(a)) for which horizontal bonds are modified from K_x to K'_x on a column (case (a)) and the chain perturbation (figure 1(b)) where vertical interactions change from K_τ to K'_τ on a column (case (b)). The magnetisation near the defect was found to vanish at the bulk critical point with a non-universal (interaction-dependent) critical exponent

$$\beta_0 = \frac{1}{2\pi^2}[\cos^{-1}(-\chi)]^2 \tag{1.1}$$

with

$$\chi = \frac{\cosh(2K_x) - \cosh(2K'_x)}{\cosh(2K_x)\cosh(2K'_x) - 1} \qquad \text{case (a)}, \tag{1.2}$$

and

$$\chi = \tanh[2(K_\tau - K'_\tau)] \qquad \text{case (b)}, \tag{1.3}$$

whereas for the chain perturbation, McCoy and Perk (1980) found a decay exponent η_0 satisfying the hyperscaling relation

$$2\beta_0 = (d - 2 + \eta_0)\nu \tag{1.4}$$

valid for any perturbation strength.

† Laboratoire Associé au CNRS No 155.

The classical Ising model in the extreme anisotropic limit ($K_x \to 0, K_\tau \to \infty$) in 2D is known to be related to the $T=0$ limit of the quantum Ising chain in a transverse field solved by Pfeuty (1970):

$$H = -\Delta \sum_m \sigma^z(m)\sigma^z(m+1) - \varepsilon \sum_m \sigma^x(m) \tag{1.5}$$

with the Suzuki equivalence (Suzuki 1971)

$$\begin{aligned} &K_x \to \Delta \\ &\exp(-2K_\tau) \to \varepsilon. \end{aligned} \tag{1.6}$$

The vertical axis in the classical problem plays the role of the time axis in the quantum problem. The equivalence may be established by using the transfer matrix formalism (Fradkin and Susskind 1978). A perturbed bond in the quantum chain corresponds to the ladder perturbation (figure 1(a)) and a local perturbation in the transverse field corresponds to the chain perturbation (figure 1(b)). This quantum problem has been studied recently by Uzelac *et al* (1981) using a real space renormalisation group approach.

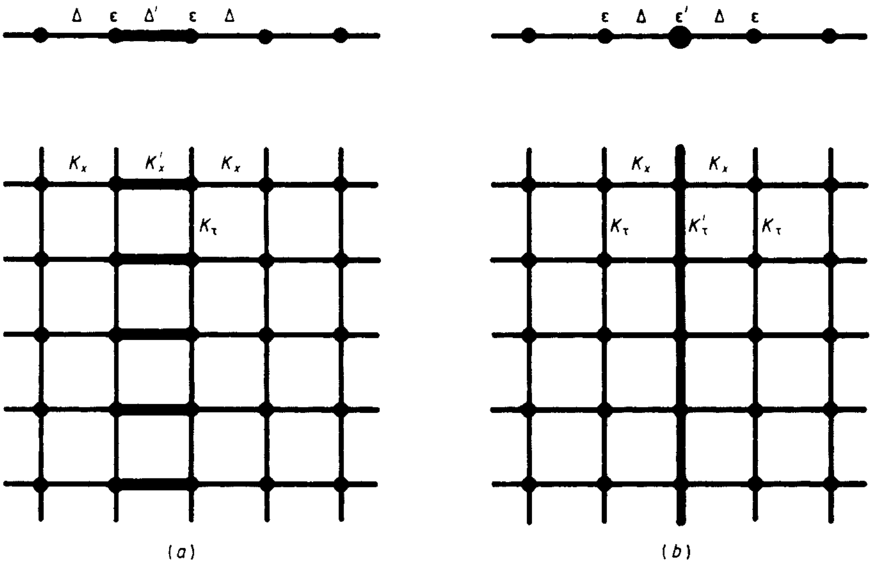

Figure 1. Quantum Ising chain in a transverse field (upper part) and its 2D classical counterpart (lower part) in the case: (a) of a ladder perturbation; and (b) of a chain perturbation.

In this work we use a weak-coupling expansion of the ground-state energy of the perturbed quantum system in a local external field to get the local magnetisation (Kogut 1979). Restricted to the fourth order in $\lambda^{-1}(\lambda = \Delta/\varepsilon)$ the perturbation expansion already gives the exact critical coupling $\lambda_c^{-1} = 1$ and magnetisation exponent $\beta = \frac{1}{8}$ in the pure case. This was our main motivation to use it in this work.

In § 2 we present the weak-coupling results for the local magnetisation in the ladder and chain problems. The last section (§ 3) is devoted to a discussion of the duality relation between the two types of perturbation.

2. Weak-coupling expansion for the local magnetisation

The 1D quantum Hamiltonian for the ladder perturbation (case (a)) reads:

$$H^{\mathrm{L}} = -\Delta \sum_{m\neq 0} \sigma^z(m)\sigma^z(m+1) - \Delta'\sigma^z(0)\sigma^z(1) - \varepsilon \sum_m \sigma^x(m) \tag{2.1}$$

where σ^z and σ^x are Pauli spin operators. To put equation (2.1) into a form which is appropriate for the weak-coupling expansion of the ground-state energy, we introduce the operator $W^{\mathrm{L}} = H^{\mathrm{L}}/\Delta$ and add a local external field h_0 in the z direction acting on site 0 so that:

$$W^{\mathrm{L}} = W_0^{\mathrm{L}} - \lambda^{-1} V^{\mathrm{L}} \tag{2.2}$$

with

$$W_0^{\mathrm{L}} = \sum_{m\neq 0} [1 - \sigma^z(m)\sigma^z(m+1)] + x[1 - \sigma^z(0)\sigma^z(1)] - h_0\sigma^z(0) \tag{2.3}$$

$$V^{\mathrm{L}} = \sum_m \sigma^x(m) \tag{2.4}$$

where $x = \Delta'/\Delta$ and an unimportant constant term has been added in equation (2.3).

For the chain perturbation (case (b)):

$$H^{\mathrm{c}} = -\Delta \sum_m \sigma^z(m)\sigma^z(m+1) - \varepsilon \sum_{m\neq 0} \sigma^x(m) - \varepsilon'\sigma^x(0) \tag{2.5}$$

so that, proceeding as above, we have:

$$W_0^{\mathrm{c}} = \sum_m [1 - \sigma^z(m)\sigma^z(m+1)] - h_0\sigma^z(0) \tag{2.6}$$

$$V^{\mathrm{c}} = \sum_{m\neq 0} \sigma^x(m) + x\sigma^x(0) \tag{2.7}$$

and $x = \varepsilon'/\varepsilon$.

The ground-state energy $E_0(h_0, x)$ is obtained through a perturbation expansion up to terms of the fourth order in λ^{-1} (see Kogut 1979) and use may be made of the Feynman–Hellmann theorem to get the local magnetisation:

$$M(x) = -\left.\frac{\partial E_0(h_0, x)}{\partial h_0}\right|_{h_0=0} = \sum_n b_n \lambda^{-n}. \tag{2.8}$$

The expansion coefficients are:

case (a)

$$\begin{aligned} b_{2n+1} &= 0 \\ b_0 &= 1 \\ b_2 &= -\frac{1}{2(1+x)^2} \\ b_4 &= \frac{1}{8(1+x)^2}\left[\frac{3}{(1+x)^2} - \frac{3(x+9)}{4(1+x)} - 1\right] + \frac{3}{8(3+x)^2}\left[\frac{7+3x}{(1+x)^3} + \frac{2(2+x)}{(1+x)^2} + \frac{1}{4}\right] \end{aligned} \tag{2.9}$$

case (b)

$$\begin{aligned} b_{2n+1} &= 0 \\ b_0 &= 1 \\ b_2 &= -x^2/8 \\ b_4 &= x^2(3x^2 - 10)/128. \end{aligned} \tag{2.10}$$

The critical exponent β_0 and the critical coupling λ_c^{-1} are obtained through a Padé approximants analysis of $\partial \ln M(x)/\partial(\lambda^{-1})$. $P(2, 2)$, $P(1, 2)$ and $P(1, 3)$ approximants give the same values. These are shown in figure 2 and figure 3 as functions of x and x^{-1} for cases (a) and (b) together with the exact results of Bariev in the Suzuki limit (equation (1.6)) where

$$\beta_0 = \frac{1}{2\pi^2}\left[\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)\right]^2 \tag{2.11}$$

for case (a), whereas x has to be changed in x^{-1} in case (b).

In case (a) (figure 2) we get exact results when $x = 1$ (unperturbed Ising chain) and when $x = 0$ (surface magnetisation). A good agreement with Bariev's results is obtained over the whole range of x values. In case (b) the perturbation expansion appears to break down around $x^{-1} \sim 0.5$. From equation (2.10), it is clear that for large x values, $b_n \sim x^n$; on the other hand, from the weak-coupling expansion on the q-state Potts model (Turban 1981) one knows that $b_n \sim q^{-n}$. It follows that the weak-coupling expansion breaks down when $x/q \sim 1$, or, with $q = 2$ (Ising), when $x^{-1} \sim 0.5$ so that going further in the expansion would not help.

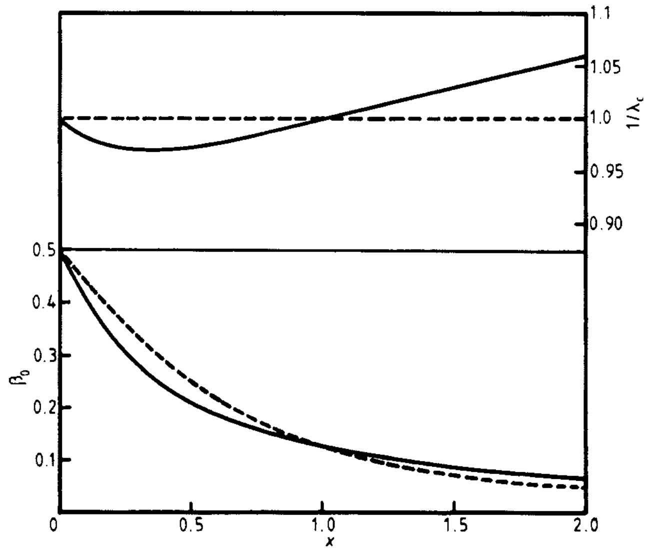

Figure 2. Exponent β_0 of the local magnetisation (lower part) and critical coupling λ_c^{-1} (upper part) deduced from the Padé approximant analysis of the weak-coupling series (full curve) and the exact result of equation (2.11) (broken curve) as functions of $x = \Delta'/\Delta$ (case (a)).

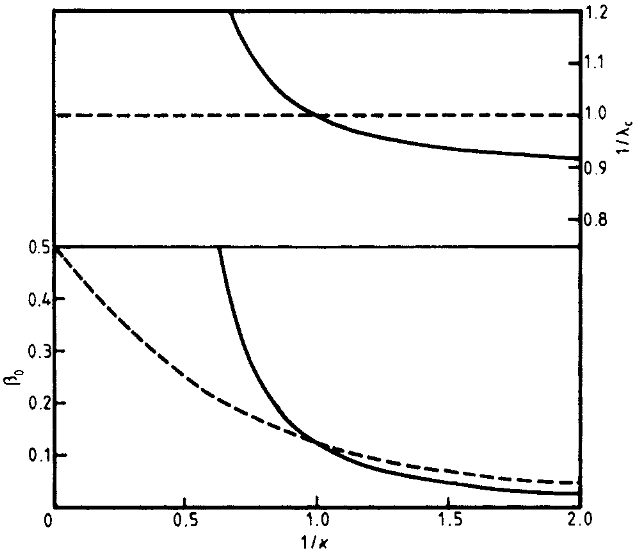

Figure 3. Same curves as in figure 2 for the chain perturbation (case (b)) with $x = \varepsilon'/\varepsilon$.

The local magnetisation may be deduced from the Padé approximants using

$$M(\lambda^{-1}, x) = \exp\left[\int_0^{\lambda^{-1}} P(N, D)(x, \lambda'^{-1})\, \mathrm{d}(\lambda'^{-1})\right]. \tag{2.12}$$

The results are given in figure 4 for case (a) and figure 5 for case (b).

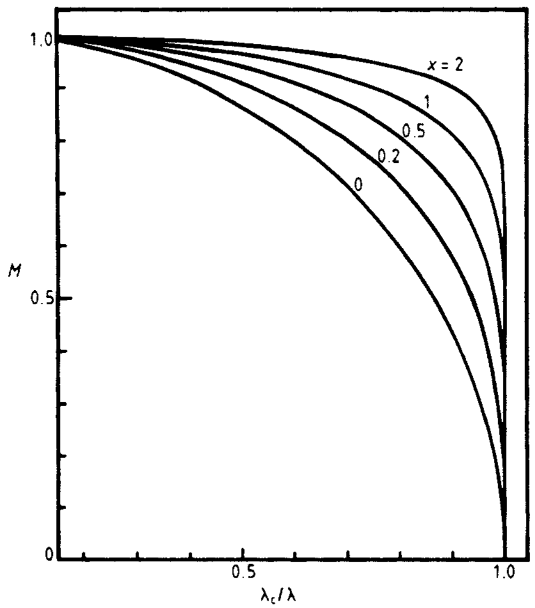

Figure 4. Local magnetisation $M = \langle\sigma^z(0)\rangle$ for the ladder perturbation as a function of the normalised coupling λ_c/λ for different values of $x = \Delta'/\Delta$.

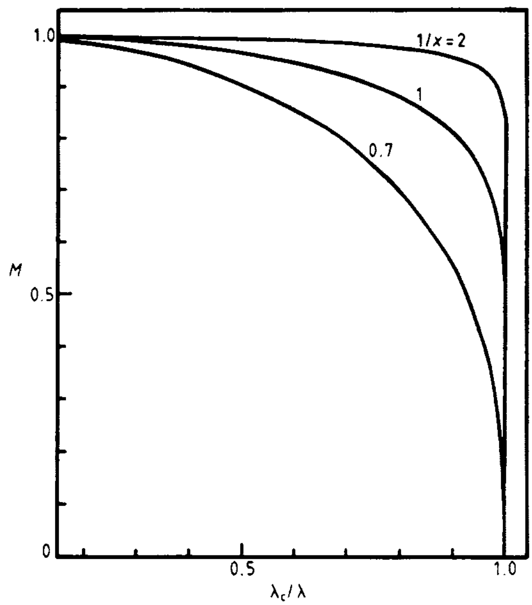

Figure 5. Local magnetisation $M = \langle\sigma^z(0)\rangle$ for the chain perturbation as a function of the normalised coupling λ_c/λ for different values of $x = \varepsilon'/\varepsilon$.

As for the unperturbed Ising chain (Kogut 1979) the weak-coupling expansion, even limited to the fourth order in λ^{-1}, gives quite reliable results for the local magnetisation. At least in case (a) the agreement with Bariev's results is better than in the renormalisation group approach of Uzelac *et al* (1981) where $n_L = 4$ levels were retained in blocks of $n_S = 3$ spins.

3. Duality

We shall end with a discussion of the duality relation between the ladder and chain perturbations which does not seem to have been explored previously. Consider the chain and ladder Hamiltonians

$$H^C(\sigma; \lambda, x) = \frac{H^C}{\Delta} = -\sum_m \sigma^z(m)\sigma^z(m+1) - \frac{1}{\lambda}\sum_{m\neq 0}\sigma^x(m) - \frac{x}{\lambda}\sigma^x(0) \tag{3.1}$$

$$H^L(\sigma; \lambda, x) = \frac{H^L}{\Delta} = -\sum_{m\neq 0}\sigma^z(m-1)\sigma^z(m) - x\sigma^z(-1)\sigma^z(0) - \frac{1}{\lambda}\sum_m \sigma^x(m). \tag{3.2}$$

Applying the duality transformation (Fradkin and Susskind 1978):

$$\mu^x(m) = \sigma^z(m)\sigma^z(m+1)$$

$$\mu^z(m) = \prod_{n\leq m}\sigma^x(n) \tag{3.3}$$

to $H^{C}(\sigma; \lambda, x)$ with a local external field h_0 acting on site 0, we get

$$H^{C}(\sigma; \lambda, x) - h_0\sigma^z(0)$$
$$= -\sum_m \mu^x(m) - \frac{1}{\lambda}\sum_{m\neq 0} \mu^z(m-1)\mu^z(m) - \frac{x}{\lambda}\mu^z(-1)\mu^z(0) - h_0 \prod_{n<0} \mu^x(n)$$
$$= \lambda^{-1} H^{L}(\mu; \lambda^{-1}, x) - h_0 \prod_{n<0} \mu^x(n), \tag{3.4}$$

so that the ground-state energies satisfy $E_0^C(\lambda) = \lambda^{-1} E_0^L(\lambda^{-1})$ and first-order perturbation theory in h_0 implies

$$\langle 0|\sigma^z(0)|0\rangle_{C,\lambda} = \langle 0| \prod_{n<0} \mu^x(n)|0\rangle_{L,\lambda^{-1}} \tag{3.5}$$

where $|0\rangle_{C,\lambda}$ is the vacuum of $H^{C}(\sigma; \lambda, x)$ and $|0\rangle_{L,\lambda^{-1}}$ the vacuum of $H^{L}(\mu; \lambda^{-1}, x)$. It follows from equation (3.5) that the 'local' disorder parameter $\Pi_{n<0}\,\mu^x(n)$ of the ladder model, which has a non-vanishing expectation value in the disordered phase, vanishes at the critical coupling with a critical exponent β_0^D which is the same as the local magnetisation exponent of the chain model and consequently differs from the local order parameter of the ladder model. To sum up, using

$$\beta_{0\,\text{chain}}(x) = \beta_{0\,\text{ladder}}(x^{-1})$$

we get the following relations between local exponents:

$$\beta^D_{0\,\text{chain}}(x) = \beta_{0\,\text{chain}}(x^{-1}) \tag{3.6}$$
$$\beta^D_{0\,\text{ladder}}(x) = \beta_{0\,\text{ladder}}(x^{-1}) \tag{3.7}$$

where the index D stands for the 'local' disorder parameter exponents. Similar relations may be written for the classical problem.

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